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NUMERICAL METHOD FOR THE SOLUTION OF PROBLEMS OF NONSTEADY
NONLINEAR THERMAL CONDUCTION OF COMPLEX TWO-DIMENSIONAL BODIES

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The method involves the use of a nonrectangular orthogonal grid, the form of which is determined by the boundary of the body; a single calculation algorithm suitable for bodies of various shapes is obtained.

Using the well-known finite-difference method [1], it is difficult to construct a single algorithm for the temperature field of two-dimensional bodies of various shapes; this is because a rectangular grid, as a rule, provides a poor fit to the boundaries of the body and requires individual programming for each body.

In physical terms, the essence of the phenomenon is that heat fluxes flow from the boundary into the body and meet at centers or lines of symmetry of the body; accordingly, consider a system of coordinate lines y in the form of straight lines from the boundary running

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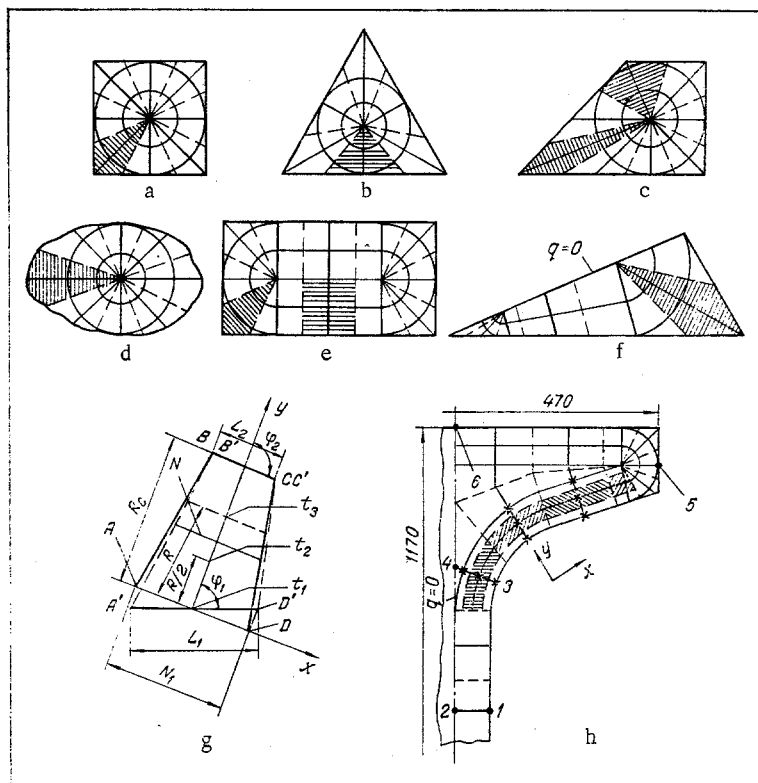


Fig. 1. Division of bodies into elements and calculational scheme for an element (g).

into the body, and another system of coordinate lines x normal to the first set, consisting of segments of straight lines and arcs of circles. For a given class of bodies it is then possible to obtain a set of standard elements, the thermal fields of which conform to a single calculation algorithm.

The principles of construction of such a grid are best explained using examples (Fig. 1).

A body whose dimensions in various directions are not greatly different, which is heated (or cooled) in all directions, has a cylindrical system of coordinates, the origin of which — a pole — is approximately at the center of mass of the body (Fig. 1a-d); when a rectangle is heated in all directions (Fig. 1e), the body has a mixed cylindrical and rectangular system of coordinates with two poles. A wedge (Fig. 1f) has two poles on the insulated side ($q = 0$), each pole serving as the origin of a cylindrical coordinate system. In Fig. 1a-f the boundaries of the elements are shown by dashed lines, and some of the elements are shaded. Such an orthogonal grid may also be constructed for bodies as complex as those encountered in practice (see, for example, the blank for preparing an I-beam — Fig. 1h).

The following rules are adopted to standardize the calculational process for dividing a body into elements:

1) each of the elements into which the body is divided must have at least one external boundary, coinciding with the boundary of the body, and one internal boundary (which may be of zero length) approximately coinciding with a line of symmetry of the body or a line of zero heat flux (if known); the straight line joining the midpoint of the external and in many cases the internal boundary of the element is called the radius;

2) three transverse lines x intersect each radius.

Now consider the thermal-conduction equation in a curvilinear orthogonal system of coordinates x, y [when $dx = dx(x, y)$ and $dy = dy(x, y)$]:

$$\rho c \frac{\partial t}{\partial \tau} dx dy = \frac{\partial}{\partial x} \left(dy \lambda \frac{\partial t}{\partial x} \right) dx + \frac{\partial}{\partial y} \left(dx \lambda \frac{\partial t}{\partial y} \right) dy, \quad (1)$$

where λ , ρ , and c are functions of the temperature. In order to use the method of alternating directions, this equation is separated into two one-dimensional equations [2]

$$\rho c \frac{\partial t}{\partial \tau} dx = \frac{\partial}{\partial y} \left(dx \lambda \frac{\partial t}{\partial y} \right), \quad (2)$$

$$\rho c \frac{\partial t}{\partial \tau} dy = \frac{\partial}{\partial x} \left(dy \lambda \frac{\partial t}{\partial x} \right), \quad (3)$$

which are solved successively for all the elements: first in the direction y - Eq. (2) - and then in the direction x - Eq. (3). The initial condition is that, when $\tau = 0$, $t(x, y) = t_0$. One boundary condition for Eq. (2) is that when $y = 0$, $\lambda dt/dy = \alpha(t - t_{\text{circ}}) + q$, where α , t_{circ} , and q are given functions of the coordinate and time; at the internal boundary the matching conditions for two elements are satisfied (equality of temperature and heat fluxes).

For Eq. (3) the matching condition for the first and second elements is satisfied if the line x is closed inside the body or the condition $\lambda dt/dx = 0$ if the element is bounded by a heat-flux symmetry line.

The integral-balance method is used to solve Eq. (2). The calculation scheme of this method for the i -th element A'B'C'D' is shown in Fig. 1g.

The coordinate origin is chosen at the external boundary of the element, coinciding with the boundary of the body; the y axis runs along the radius of the element (into the body). Drawing from the points A'B'C'D' straight lines parallel to the radius that intersect with the normals to the radius at the points $y = 0$ and $y = R_c$, we obtain a new calculational element ABCD, practically equivalent in area to the original. Introducing [3] the thickness of heating (cooling) element R (a running radius, beyond which the temperature still has its initial value), the temperature field of the heated (cooled) part of the element may be characterized by a parabola passing through three points with coordinates $y = 0$, $0.5R$, and R and corresponding temperatures t_1 , t_2 , and t_3 :

$$t = t_j a_{g,j} (y/R)^g = t_j a_{g,j} \eta^g, \quad (4)$$

where $\eta = y/R$. This formula, and those below, employs the convention of tensor analysis by which the summation sign is omitted if summation over repeating indices is implied. Here $a_{g,j}$ denotes the following set of nine coefficients: $\alpha_{0,1} = 1$, $\alpha_{0,2} = 0$, $\alpha_{0,3} = 0$, $\alpha_{1,1} = -3$, $\alpha_{1,2} = 4$, $\alpha_{1,3} = -1$, $\alpha_{2,1} = 2$, $\alpha_{2,2} = -4$, and $\alpha_{2,3} = 2$.

Substituting I for t , an expression analogous to Eq. (4) can be written for the enthalpy.

The running transverse dimension of the element dx (along the y axis) is defined as the value of the normal to the radius depending linearly on y

$$dx \equiv N = N_1 + N_2 y = N_m y^{m-1} \quad (m = 1, 2), \quad (5)$$

where

$$N_1 = L_1 \sin \varphi_1; \quad N_2 = (L_2 \sin \varphi_2 - L_1 \sin \varphi_1)/R_c.$$

It is simple to establish that

$$\int_{y_1}^{y_2} N dy = N_m \frac{y^m}{m} \Big|_{y_1}^{y_2} \quad (5a)$$

gives the area of the part of the element included between the coordinates y_1 and y_2 . Integrating Eq. (2) between y and R gives

$$\int_y^R \frac{\partial I}{\partial \tau} N dy = N \lambda \frac{\partial t}{\partial y} \Big|_y^R. \quad (6)$$

If t and I in the form in Eq. (4) and N in the form in Eq. (5) are substituted into Eq. (6), the result is

$$(1 - \eta^{g+m}) a_{g,j} N_m R^m \frac{\partial I_j}{\partial \tau} \cdot \frac{1}{(g+m)} - (1 - \eta^{g+m}) a_{g,j} N_m R^{m-1} I_j \times \frac{\partial R}{\partial \tau} \cdot \frac{1}{(g+m)} = (\lambda_R - \lambda_y \eta^{g+m-2}) t_j g a_{g,j} N_m / R, \quad (7)$$

$$g = 0, 1, 2; j = 1, 2, 3; m = 1, 2.$$

Taking the values $\eta_1 = 0.5$, $\eta_2 = 0.25$, and $\eta_3 = 0$ obtained from preliminary calculations, Eq. (7) gives a system of three equations:

$$B_{s,j} \Delta I_j + \bar{B}_s \Delta R = C_s, \quad (8)$$

where

$$\left. \begin{aligned} C_s &= (d'_{s,j}/2 - d_{s,j}) t_j \Delta \tau - d'_{s,j}/2 \cdot \bar{t}_j \Delta \tau \quad (s = 1, 2); \\ B_{s,j} &= (1 - \eta_s^{g+m}) a_{g,j} N_m R^m / (g+m) + \Delta B_{s,j} \\ \bar{B}_s &= -(1 - \eta_s^{g+m}) a_{g,j} g N_m R^{m-1} I_j / (g+m) \\ d'_{s,j} &= \lambda_{\eta=1} g a_{g,j} N_m / R \\ d_{s,j} &= \lambda_{\eta=\eta_s} g a_{g,j} N_m \eta_s^{g+m-2} / R \\ \Delta B_{s,j} &= -(d'_{s,j} - d_{s,j}) \Delta \tau / 2 \end{aligned} \right\}, \quad (9)$$

and \bar{t}_j is the temperature in an element having a common internal boundary L_2 with the given element. At the external boundary of the element of length L_1 ($s = 3$), the boundary condition is satisfied, i.e., C_3 is calculated from the formula

$$C_3 = [\alpha (t_1 - t_{\text{circ}}) + q] L_1 \Delta \tau.$$

The increments $\Delta B_{s,j} = \partial C_s / \partial t_j$ to $B_{s,j}$ in Eq. (9) arise because the coefficients C_s are not taken at the moment τ , but at the midpoint of a time interval, i.e., at the moment $\tau + \Delta \tau / 2$ at temperatures $t_j + \Delta t_j / 2$; in fact,

$$\Delta B_{3,2} = \Delta B_{3,3} = 0; \quad \Delta B_{3,1} = \partial C_3 / \partial t_1.$$

In this case the stability of the system is increased by an order of magnitude.

At the first stage of heating, Eq. (8) is solved for the unknowns R , t_1 , and t_2 (t_3 is equal to the initial temperature).

At the second stage, when $R = R_C$, Eq. (8) is solved for the unknowns t_1 , t_2 , and t_3 .

When the temperature of all the elements at the given moment of time has been calculated from Eqs. (8) and (9), i.e., the solution of Eq. (2) in the direction y is finished, Eq. (3) is solved in the direction x .

To do so, the volume of each element is divided into three parts [see Fig. 1h, where the shading indicates the middle part of the volumes, and the crosses denote points of intersection of radial (y) and transverse (x) lines].

In calculating the volume of each part it is noted that the boundaries of the middle part run midway between points of intersection of the radial and transverse lines.

We now have three series of curvilinear volumes, corresponding to the number of transverse lines x . It is possible to solve Eq. (3) by an implicit method, for example, the Crank-Nicholson method, but adequate accuracy will only be obtained if $\Delta \tau < k(\Delta x)^2/a$, where, as found in carrying out the present calculations and also in [4], $k < 2.5$. Since, in our grid, sections with small Δx may be encountered, the calculation time will be high. The time step can be increased by the following method; the appropriate calculation scheme for the series considered (indicated by the shading in Fig. 1h) is shown in Fig. 2.

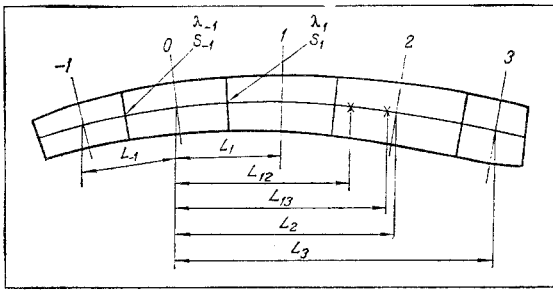


Fig. 2. Calculation of transverse overflow between elements.

First consider the thermal interaction of only two volumes V_0 and V_1 with a common boundary s_1 and a distance l_1 between their centers. In this case, Eq. (3) corresponds to the following expression (noting that l_1 corresponds to dx and s_1 to dy):

$$dt_0 = \frac{a_1}{V_0} \cdot \frac{s_1}{l_1} (t_1 - t_0) d\tau, \quad (10)$$

which may be written in the form

$$dt_0 = \frac{a_1}{V_0} \cdot \frac{s_1}{l_1} \cdot \frac{(W_0 + W_1)}{W_1} (t_s - t_0) d\tau, \quad (11)$$

where

$$t_s = (Q_0 + Q_1)/(W_0 + W_1); \quad Q_i = \rho_i C_i V_i t_i; \quad W_i = \rho_i C_i V_i. \quad (12)$$

Since the mean temperature t_s of an isolated system of two volumes is constant over time, integrating Eq. (11) gives, for any $\Delta\tau$, the result

$$\Delta t_0 = C_1 (t_1 - t_0) [1 - \exp(-\mu_1 \Delta\tau)]; \quad (13)$$

$$C_1 = W_1/(W_0 + W_1), \quad \mu_1 = a_1 s_1 / (l_1 V_0 C_1). \quad (14)$$

Now consider separately the interaction of a volume V_0 and $V_{12} = V_1 + V_2$; assume that the temperature of V_{12} is

$$t_{12} = (Q_1 + Q_2)/(W_1 + W_2).$$

In this case,

$$\begin{aligned} \Delta t_0 &= C_{12} (t_{12} - t_0) [1 - \exp(-\mu_2 \Delta\tau)] \\ &= C_{12}/(W_1 + W_2) \cdot [W_1 (t_1 - t_0) + W_2 (t_2 - t_0)] [1 - \exp(-\mu_2 \Delta\tau)], \end{aligned} \quad (15)$$

$$\mu_2 = a_1 s_1 / (l_{12} V_0 C_{12}), \quad \mu_i = a_1 s_1 / (l_i V_0 C_{1i}), \quad (16)$$

and l_{12} is the distance between the centers of the volumes V_0 and V_{12}

$$\begin{aligned} l_{12} &= (l_1 V_1 + l_2 V_2)/(V_1 + V_2), \quad C_{12} = (W_1 + W_2)/(W_0 + W_1 + W_2), \\ l_{1i} &= (l_1 V_1 + l_2 V_2 + \dots + l_i V_i)/(V_1 + V_2 + \dots + V_i); \\ C_{1i} &= (W_1 + W_2 + \dots + W_i)/(W_0 + W_1 + \dots + W_i). \end{aligned}$$

Now assume that the simultaneous interaction of two pairs of volumes (V_0 and V_1) and (V_0 and V_{12}) may be written in the form

$$\begin{aligned} \Delta t_0 &= [k_{1,1} (t_1 - t_0) + k_{1,2} (t_2 - t_0)] [1 - \exp(-\mu_1 \Delta\tau)] \\ &\quad + [k_{2,1} (t_1 - t_0) + k_{2,2} (t_2 - t_0)] [1 - \exp(-\mu_2 \Delta\tau)], \end{aligned} \quad (17)$$

where μ_1 and μ_2 are given by Eqs. (14) and (16). By analogy, the formula for the interaction of the volume V_0 with n volumes lying to the right of V_0 is

$$\begin{aligned} \Delta t_{0,R} &= [k_{1,1} (t_1 - t_0) + k_{1,2} (t_2 - t_0)] [1 - \exp(-\mu_1 \Delta\tau)] \\ &\quad + [k_{2,1} (t_1 - t_0) + k_{2,2} (t_2 - t_0) + k_{2,3} (t_3 - t_0)] [1 - \exp(-\mu_2 \Delta\tau)] + \end{aligned}$$

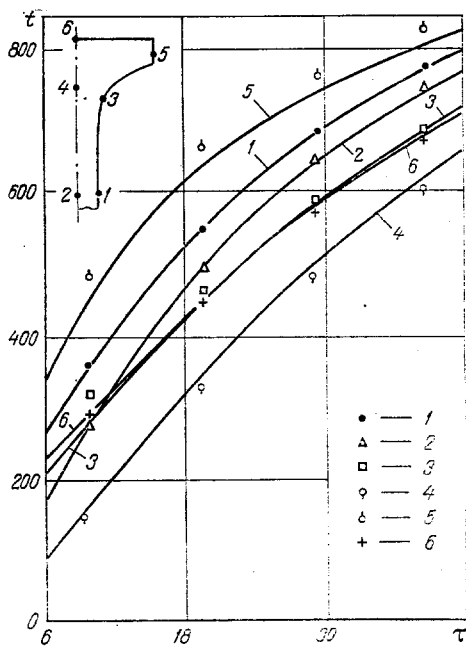


Fig. 3. Temperature curves for the heating of an I-beam (Fig. 1h): the continuous curves show results given by the new method; the points give results obtained by the method of [5]. t , °C; τ , min.

$$+ [k_{3,3}(t_3 - t_0) + k_{3,4}(t_4 - t_0)] [1 - \exp(-\mu_3 \Delta\tau)] + \dots \\ + [k_{i,i}(t_i - t_0) + k_{i,i+1}(t_{i+1} - t_0)] [1 - \exp(-\mu_i \Delta\tau)] + \dots + [k_{n,n}(t_n - t_0)] [1 - \exp(-\mu_n \Delta\tau)]. \quad (18)$$

The number n is chosen according to the inequality $l_n > 1.5\sqrt{a\Delta\tau}$, where l_n is the distance from V_0 to V_n ; in our present calculations, $n = 1-3$.

Analogously, an expression can be written for the thermal interaction of the volume V_0 with m volumes lying to the left, with negative subscripts from -1 to $-m$.

The total change of temperature in the volume V_0 is

$$\Delta t_0 = \Delta t_{0,R} + \Delta t_{0,L}. \quad (19)$$

The coefficients $k_{i,j}$ are determined from the limiting conditions: as $\Delta\tau \rightarrow 0$

$$\Delta t_0 = \left[\frac{a_1}{V_0} \cdot \frac{s_1}{l_1} (t_1 - t_0) + \frac{a_{-1}}{V_0} \cdot \frac{s_{-1}}{l_{-1}} (t_{-1} - t_0) \right] \Delta\tau, \quad (20)$$

as $\Delta\tau \rightarrow \infty$, t_0 tends to the mean temperature t_s of all the volumes, i.e.,

$$(t_s - t_0) \rightarrow \sum_{-m}^n (t_i - t_0) W_i / \left(\sum_{-m}^n W_i \right).$$

These conditions give the following pairs of equations:

as $\Delta\tau \rightarrow 0$

$$k_{i,i} \mu_i + k_{i+z,i} \mu_{i+z} = a_i s_i / (l_i V_0),$$

as $\Delta\tau \rightarrow \infty$

$$k_{i,i} + k_{i+z,i} = W_i / \left(\sum_{-m}^n W_i \right),$$

as $\Delta\tau \rightarrow 0$

$$k_{i,i} \mu_i + k_{i-z,i} \mu_{i-z} = 0,$$

as $\Delta\tau \rightarrow \infty$

$$k_{i,i} + k_{i-z,i} = W_i / \left(\sum_{-m}^n W_i \right),$$

for $i = \pm 1$,

for $i \neq \pm 1$,

where $z = \text{sign}(i)$.

Calculation by Eqs. (18) and (19) is convenient, because it is absolutely stable and sufficiently accurate for a time step far larger than that of implicit methods such as the Crank-Nicholson method.

This method of calculation gives results that are in fair agreement with the known accurate solutions for simple bodies [1]: the mean-square error does not exceed 0.1-0.2% for a cylinder and a plate (over three temperatures) and 0.2-0.4% for a square (over 24 points) for a Biot number less than unity and does not exceed 1-2% and 2-4%, respectively, for a Biot number higher than 10. On a Minsk-22 computer the calculation of one temperature for one element over one time step took 2-3 sec. For a square of eight elements, dividing the total heating interval into 20-30 steps, the calculation required 8-10 min and provided the accuracy given above.

In Fig. 3 results obtained using the new method and the finite-difference method (the latter obtained by V. M. Malkin at the All-Union Scientific-Research Institute of Metallurgical Thermoengineering using the scheme given in [5]) are compared for the heating of the I-beam in Fig. 1h under the following conditions: $\lambda = 29.1 \text{ W/(m}\cdot\text{deg)}$, $c = 0.688 \text{ kJ/(kg}\cdot\text{deg)}$, $\rho = 7800 \text{ kg/m}^3$, ambient temperature 1000°C , initial temperature of body 0°C , and heat-transfer coefficient $163 \text{ W/(m}^2\cdot\text{deg)}$. As is evident, the two methods give approximately the same results, while the new method is approximately twice as fast as the finite-difference method.

The major advantage of the method outlined in the present paper is that in passing from one body to another it is necessary to change only the initial data (as shown by experience in carrying out calculations, this takes no more than 6-8 h, including construction of the grid), retaining the program unchanged.

NOTATION

x, y , coordinates (x is transverse to the radius; y is directed into the element along the radius); R , thickness of heating (cooling) of the element; R_C , longitudinal dimension of element; L_1, L_2 , external and internal boundaries of element; $\eta = y/R$, dimensionless coordinate; l_n , distance between the given element and its neighbor; φ_1, φ_2 , angles between L_1, L_2 and R_C ; N , running transverse dimension of element; V , volume; s_1, s_{-1} , length of line of contact of the given element with its neighbor to the right and to the left; t , temperature; I , enthalpy; a , thermal conductivity ($a_1 = \lambda_1/\rho_0 c_0$; $a_{-1} = \lambda_{-1}/\rho_0 c_0$); λ , thermal conductivity; $Q_i = \rho_i c_i V_i t_i$; $W_i = \rho_i c_i V_i$.

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